

On Past Singularities in $k = 0$ FLRW Cosmologies

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The fundamental singularity theorem of FLRW cosmologies assumes that the matter content in the cosmological model obeys the strong energy condition along with a nonpositive cosmological constant which gives rise to an irrotational geodesic singularity. In this paper, we show that the important case of a spatially flat Friedmann-Lemaître-Robertson-Walker universe with barotropic matter obeying only the *weak* energy condition with a nonnegative cosmological constant also contains a past singularity. We accomplish this using topological methods from dynamical systems theory. The methods employed in this paper are sufficiently general that they could be extended to other models to study the existence of past singularities.

I. INTRODUCTION

Some cosmological solutions to the Einstein field equations are understood to begin at a certain cosmic time, which for the sake of simplicity is usually denoted as time $t = 0$, which is usually referred to as the *point of time of the Big Bang* [1]. The issue of singularities is important in the context of cosmology because this can possibly give us an answer to whether the Big Bang actually occurred or not.

If one considers a cosmological constant denoted by Λ and denotes the energy density and pressure of the matter in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe by μ and p respectively, then the fundamental singularity theorem for FLRW cosmologies states that if $\Lambda \leq 0$, $\mu + 3p \geq 0$, and $\mu + p > 0$ (that is, the matter obeys the strong energy condition) in a fluid flow for which there is no acceleration or rotation, then an irrotational geodesic singularity will occur at a finite proper time [2–5].

With respect to Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes, the concept of initial singularities / big bang singularities has been studied a number of times in the literature in various contexts from classical general relativity to quantum cosmology [6–45]. Further, the classic texts [46, 47] describe the singularity theorems in significant detail.

In this paper, we consider a $k = 0$ FLRW cosmology with barotropic matter, where $p = w\mu$, and a positive cosmological constant, $\Lambda > 0$. We demonstrate that assuming only the weak energy condition (WEC), that all such models are past asymptotic to a big bang singularity, thereby extending the classic singularity theorem as described above. Note that, throughout, we use units where $8\pi G = c = 1$.

II. THE DYNAMICAL EQUATIONS

Following [1, 48], we consider an orthonormal frame approach, where one can reduce the Einstein field equations to a system of first-order, autonomous dynamical equations. Namely, one obtains the Raychaudhuri equation,

$$\dot{\theta} + \frac{1}{3}\theta^2 + \frac{1}{2}(\mu + 3p) - \Lambda = 0, \quad (1)$$

the Friedmann equation,

$$\frac{1}{3}\theta^2 = \mu + \Lambda, \quad (2)$$

and the energy-momentum conservation equation,

$$\dot{\mu} + \theta(\mu + p) = 0. \quad (3)$$

Now, the advantage to considering $k = 0$ FLRW models, is that one can use the Friedmann equation (2) to eliminate μ from the Raychaudhuri equation (1), to obtain a single dynamical equation (after employing a barotropic equation of state, $p = w\mu$):

$$\dot{\theta} = -\frac{1}{2}(w + 1)(\theta^2 - 3\Lambda). \quad (4)$$

The big bang singularity occurs for when $\theta = \infty$. It is of interest to determine whether $\theta \rightarrow \infty$ is indeed a past asymptotic state of such a model. We wish to compactify this infinity by employing the following coordinate transformation, which takes place in the phase space of our ordinary differential equation. Namely, we let

$$X = \frac{1}{1 + \exp(-\theta)}. \quad (5)$$

Therefore, we have that

$$\lim_{\theta \rightarrow \infty} X = 1. \quad (6)$$

Therefore, the singularity $\theta = \infty$ appears as $X = 1$ in our transformed variable.

Further, under the transformation Eq. (5), the Raychaudhuri equation (4) takes the form

$$\dot{X} = -\frac{1}{2}(1 + w)(X - 1)X \left[3\Lambda - \log^2 \left(-1 + \frac{1}{X} \right) \right]. \quad (7)$$

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III. THE SINGULARITY AS A SOURCE OF RAYCHAUDHURI'S EQUATION

With the Raychaudhuri equation (7) in hand, we are in a position to describe its fixed points and behaviour with respect to these points. In actuality, we are only interested in the point $X = 1$ which corresponds to $\theta = \infty$ in the original coordinates. To prove the existence of a big bang singularity, our goal is to show that the point $X = 1$ is a past asymptotic state of Eq. (7).

The first order of business is to determine whether $X = 1$ is indeed a fixed point of Eq. (7). Indeed, there may be some objection to this because the log function on the right hand side of Eq. (7) blows up for $X = 1$. However, note that

$$\lim_{X \rightarrow 1} -\frac{1}{2}(1+w)(X-1)X \left[3\Lambda - \log^2 \left(-1 + \frac{1}{X} \right) \right] = 0. \quad (8)$$

Therefore, $X = 1$ is indeed a fixed point of Eq. (7). We have compactified the big bang singularity to being a fixed point of our transformed equation.

What remains to be done is to prove that $X = 1$ is a past asymptotic state of the dynamics represented by Eq. (7). The standard methodology is to first determine local behaviour, that is, whether $X = 1$ is a source of the system by analyzing the the sign of the first derivative of the right-hand-side of Eq. (7) at $X = 1$. However, this cannot be done in this case. If we denote the right-hand-side of Eq. (7) by $f(X)$, then $f'(X)$ is given by

$$f'(X) = -\frac{1}{2}(w+1) \left[3\Lambda(2X-1) + (1-2X) \log^2 \left(\frac{1}{X} - 1 \right) - 2 \log \left(\frac{1}{X} - 1 \right) \right], \quad (9)$$

which is not defined at $X = 1$. Further,

$$\lim_{X \rightarrow 1} f'(X) = -\infty. \quad (10)$$

Therefore, a local stability analysis as suggested in [49], for example, will not work in this case.

We therefore make use of Chetaev's instability theorem to show that $X = 1$ is unstable, that is, it is a past state of Eq. (7). Following [50], we note that a differentiable function f is called a *Chetaev function* for a singular point x_0 of a vector field v if it satisfies the following conditions: the function f is defined on a domain W whose boundary contains x_0 , then $f(x) \rightarrow 0$ as $x \rightarrow x_0$, $x \in W$; $f > 0$ and $\dot{f} > 0$ everywhere in W . Then *Chetaev's instability theorem* states that a singular point for which a Chetaev function exists is unstable.

Motivated by this theorem, let us define a function Z , such that

$$Z = -\log X, \quad (11)$$

and a domain W such that

$$W = \left\{ \frac{1}{1 + \exp \left(-\sqrt{3}\sqrt{\Lambda} \right)} < X < 1 \right\}. \quad (12)$$

Clearly, the boundary of W is given by

$$\partial W = \left\{ X = \frac{1}{1 + \exp \left(-\sqrt{3}\sqrt{\Lambda} \right)} \right\} \cup \{X = 1\}, \quad (13)$$

which importantly contains the fixed point $X = 1$.

To apply Chetaev's theorem, first note that

$$\lim_{X \rightarrow 1} Z = \lim_{X \rightarrow 1} -\log X = 0. \quad (14)$$

Further, the function Z , and its derivative, $\dot{Z} = -\dot{X}/X$ are clearly strictly positive in W (which can be seen using Eq. (7)) on the condition that $-1 < w \leq 1$ and $\Lambda > 0$. For a barotropic equation of state the condition $w > -1$ is just the weak energy condition. The condition $\Lambda > 0$ indicates that we must have a positive cosmological constant.

Therefore, we have just shown that for a positive cosmological constant and assuming that the matter distribution in our model satisfies the weak energy condition, $X = 1$ is indeed a source of Eq. (7). That is, there exists a one-dimensional unstable manifold of $X = 1$ that is tangent to the one-dimensional unstable subspace at $X = 1$ such that all orbits in the unstable manifold are asymptotic to $X = 1$ as $t \rightarrow -\infty$.

What remains to be shown is whether $X = 1$ is indeed asymptotically unstable. To accomplish this, we will use the LaSalle invariance principle [49], but modified for α -limit sets. This reads as follows. Consider a dynamical system $x' = f(x)$ on \mathbb{R}^n , with flow ϕ_t . Let S be a closed, bounded, and negatively invariant set of ϕ_t , and let Z be a C^1 monotone function, where $\dot{Z} \leq 0$. Then, for all x_0 in S , $\alpha(x_0) \subseteq \{x \in S | \dot{Z} = 0\}$. Let $Z_1 = X$. Then, Z_1 is clearly monotone for $X = 1$ which is a local source by the preceding analysis, and is therefore a negatively invariant closed set. Therefore, applying the LaSalle invariance principle, we find that $\alpha(x) = \{X = 1\}$. We therefore conclude that $X = 1$ is asymptotically unstable.

In other words, since $X = 1$ represents $\theta = \infty$ in the original variable choice, we have just shown that *a $k = 0$ FLRW model with a positive cosmological constant and barotropic matter obeying only the weak energy condition is past asymptotic to a big bang singularity.*

IV. CONCLUSIONS

In this brief paper, we considered the dynamics of a spatially flat FLRW spacetime with a positive cosmological constant and matter obeying a barotropic equation of state. By performing a coordinate transformation in the configuration space on the Raychaudhuri equation in

order to compactify the big bang singularity to a finite point. The fundamental singularity theorem of FLRW cosmologies assumes that the matter content in the cosmological model obeys the strong energy condition along with a nonpositive cosmological constant which gives rise to an irrotational geodesic singularity. We showed that a spatially flat Friedmann-Lemaître-Robertson-Walker universe with barotropic matter obeying only the *weak* energy condition with a *nonnegative* cosmological constant also contains a past singularity. This was demonstrated by first showing that the big bang singularity point was locally unstable using Chetaev's instability the-

orem. We then used the LaSalle invariance principle to show that this point is in fact an asymptotically unstable equilibrium point of Raychaudhuri's equation as applied to the model under consideration.

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- [1] Ø. Grøn and S. Hervik, *Einstein's General Theory of Relativity: With Modern Applications in Cosmology*, Springer, first ed., 2007.
 - [2] G. F. Ellis, R. Maartens, and M. A. MacCallum, *Relativistic Cosmology*, Cambridge University Press, first ed., 2012.
 - [3] G. F. R. Ellis, "On the Raychaudhuri equation," *Pramana*, vol. 69, p. 15, July 2007.
 - [4] R. C. Tolman and M. Ward, "On the behavior of non-static models of the universe when the cosmological term is omitted," *Physical Review*, vol. 39, pp. 835–843, 1932.
 - [5] A. Raychaudhuri, "Relativistic cosmology. I," *Physical Review*, vol. 98, pp. 1123–1126, 1955.
 - [6] A. G. Lemaître, "Contributions to a British Association Discussion on the Evolution of the Universe.," *Nature (London)*, vol. 128, pp. 704–706, Oct. 1931.
 - [7] G. Lemaître, "The Beginning of the World from the Point of View of Quantum Theory.," *Nature (London)*, vol. 127, p. 706, May 1931.
 - [8] F. Hoyle and J. V. Narlikar, "Effect of Quantum Conditions in a Friedmann Cosmology," *Nature (London)*, vol. 228, pp. 544–545, Nov. 1970.
 - [9] G. F. R. Ellis and W. L. Roque, "The nature of the initial singularity," *General Relativity and Gravitation*, vol. 17, pp. 397–406, Apr. 1985.
 - [10] V. Mueller and H.-J. Schmidt, "On Bianchi type-I vacuum solutions in $R + R$ -squared theories of gravitation. I - The isotropic case," *General Relativity and Gravitation*, vol. 17, pp. 769–781, Aug. 1985.
 - [11] P. Spindel and P. Vandeputte, "Numerical investigations in cosmogenesis," in *Liege International Astrophysical Colloquia*, vol. 26 of *Liege International Astrophysical Colloquia*, pp. 121–131, 1986.
 - [12] I. Prigogine, J. Geheniau, E. Gunzig, and P. Nardone, "Thermodynamics of Cosmological Matter Creation," *Proceedings of the National Academy of Science*, vol. 85, pp. 7428–7432, Oct. 1988.
 - [13] R. N. Henriksen and K. Patel, "Null charts and naked singularities in spherically symmetric, homothetic spacetimes," *General Relativity and Gravitation*, vol. 23, pp. 527–581, May 1991.
 - [14] A. Rebhan, "Large-scale rotational perturbations of a Friedmann universe with collisionless matter and primordial magnetic fields," *Astrophys. J.*, vol. 392, pp. 385–393, June 1992.
 - [15] R. P. A. C. Newman, "On the Structure of Conformal Singularities in Classical General Relativity. II Evolution Equations and a Conjecture of K. P. Tod," *Proceedings of the Royal Society of London Series A*, vol. 443, pp. 493–515, Dec. 1993.
 - [16] M. A. H. MacCallum, "A class of homogeneous cosmological models III: Asymptotic behaviour," *Communications in Mathematical Physics*, vol. 20, pp. 57–84, Mar. 1971.
 - [17] S. Capozziello, R. de Ritis, and P. Scudellaro, "Nonminimal coupling and cosmic no-hair theorem," *Physics Letters A*, vol. 188, pp. 130–136, May 1994.
 - [18] Abdussattar and R. G. Vishwakarma, "A model of the universe with decaying vacuum energy.," *Pramana*, vol. 47, pp. 41–55, July 1996.
 - [19] M. Szydlowski and A. Krawiec, "Dynamical system approach to cosmological models with a varying speed of light," *Phys. Rev. D*, vol. 68, p. 063511, Sept. 2003.
 - [20] C. S. Camara, M. R. de Garcia Maia, J. C. Carvalho, and J. A. Lima, "Nonsingular FRW cosmology and nonlinear electrodynamics," *Phys. Rev. D*, vol. 69, p. 123504, June 2004.
 - [21] M. Szydlowski, W. Godłowski, A. Krawiec, and J. Golbiak, "Can the initial singularity be detected by cosmological tests?," *Phys. Rev. D*, vol. 72, p. 063504, Sept. 2005.
 - [22] V. V. Karbanovski, A. S. Tarasova, A. S. Salimova, G. V. Bilinskaya, and A. N. Sumbulov, "Model of the static universe within GR," *Soviet Journal of Experimental and Theoretical Physics*, vol. 112, pp. 60–62, Jan. 2011.
 - [23] G. Oliveira-Neto, G. A. Monerat, E. V. Corrêa Silva, C. Neves, and L. G. Ferreira Filho, "Quantization of Friedmann-Robertson-Walker Spacetimes in the Presence of a Cosmological Constant and Stiff Matter," *International Journal of Theoretical Physics*, vol. 52, pp. 2991–3006, Sept. 2013.
 - [24] I. Antoniadis, J. Rizos, and K. Tamvakis, "Singularity-free cosmological solutions of the superstring effective action," *Nuclear Physics B*, vol. 415, pp. 497–514, Mar. 1994.
 - [25] J. P. Abreu, P. Crawford, and J. P. Mimoso, "Exact conformal scalar field cosmologies," *Classical and Quantum Gravity*, vol. 11, pp. 1919–1939, Aug. 1994.
 - [26] G. A. Monerat, H. P. de Oliveira, and I. D. Soares, "Chaos in preinflationary Friedmann-Robertson-Walker universes," *Phys. Rev. D*, vol. 58, p. 063504, Sept. 1998.
 - [27] A. Coley and M. Goliath, "Closed cosmologies with a perfect fluid and a scalar field," *Phys. Rev. D*, vol. 62,

- p. 043526, Aug. 2000.
- [28] H. Ishihara, “Big bang of the brane universe,” *Phys. Rev. D*, vol. 66, p. 023513, July 2002.
 - [29] O.-C. Stoica, “Beyond the Friedmann-Lemaître-Robertson-Walker Big Bang singularity,” *ArXiv e-prints*, Mar. 2012.
 - [30] O. Cristinel Stoica, “An Exploration of the Singularities in General Relativity,” *ArXiv e-prints*, July 2012.
 - [31] C. Gao, “A Model of Nonsingular Universe,” *Entropy*, vol. 14, pp. 1296–1305, July 2012.
 - [32] J. de Haro, J. Amorós, and S. Pan, “A simple nonsingular inflationary quintessential model,” *ArXiv e-prints*, Jan. 2016.
 - [33] A. Beesham, “Comment on the Big-Bang singularity in the scale-covariant theory,” *Astrophysics and Space Science*, vol. 123, pp. 405–407, June 1986.
 - [34] J. W. Moffat and D. Vincent, “The early universe in a generalized theory of gravitation,” *Canadian Journal of Physics*, vol. 60, pp. 659–663, May 1982.
 - [35] L. Parisi, M. Bruni, R. Maartens, and K. Vandersloot, “The Einstein static universe in loop quantum cosmology,” *Classical and Quantum Gravity*, vol. 24, pp. 6243–6253, Dec. 2007.
 - [36] A. Kreienbuehl, “Singularity avoidance and time in quantum gravity,” *Phys. Rev. D*, vol. 79, p. 123509, June 2009.
 - [37] S. Alexander and T. Biswas, “Cosmological BCS mechanism and the big bang singularity,” *Phys. Rev. D*, vol. 80, p. 023501, July 2009.
 - [38] B. Vakili, “Classical and quantum dynamics of a perfect fluid scalar-metric cosmology,” *Physics Letters B*, vol. 688, pp. 129–136, May 2010.
 - [39] I. D. Lawrie, “Internal time, test clocks, and singularity resolution in dust-filled quantum cosmology,” *Phys. Rev. D*, vol. 85, p. 023512, Jan. 2012.
 - [40] T. Pawłowski and A. Ashtekar, “Positive cosmological constant in loop quantum cosmology,” *Phys. Rev. D*, vol. 85, p. 064001, Mar. 2012.
 - [41] O.-C. Stoica, “Einstein equation at singularities,” *Central European Journal of Physics*, vol. 12, pp. 123–131, Feb. 2014.
 - [42] T. Pawłowski, R. Pierini, and E. Wilson-Ewing, “Loop quantum cosmology of a radiation-dominated flat FLRW universe,” *Phys. Rev. D*, vol. 90, p. 123538, Dec. 2014.
 - [43] O. C. Stoica, “The Friedmann-Lemaître-Robertson-Walker Big Bang Singularities are Well Behaved,” *International Journal of Theoretical Physics*, vol. 55, pp. 71–80, Jan. 2016.
 - [44] M. V. Battisti and G. Montani, “The Big-Bang singularity in the framework of a Generalized Uncertainty Principle,” *Physics Letters B*, vol. 656, pp. 96–101, Nov. 2007.
 - [45] G. F. R. Ellis and R. Maartens, “The emergent universe: inflationary cosmology with no singularity,” *Classical and Quantum Gravity*, vol. 21, pp. 223–232, Jan. 2004.
 - [46] S. Hawking and G. Ellis, *The large scale structure of space-time*. Cambridge University Press, twentieth printing ed., 2006.
 - [47] R. M. Wald, *General Relativity*. University of Chicago Press, 1984 ed., 1984.
 - [48] G. Ellis and M. MacCallum, “A class of homogeneous cosmological models,” *Comm. Math. Phys.*, vol. 12, pp. 108–141, 1969.
 - [49] J. Wainwright and G. Ellis, *Dynamical Systems in Cosmology*. Cambridge University Press, first ed., 1997.
 - [50] D. Anosov, S. K. Aranson, V. Arnold, I. Bronshtein, V. Grines, and Y. Il’yashenko, *Ordinary Differential Equations and Smooth Dynamical Systems*. Springer-Verlag, third ed., 1997.